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Dissipative Abelian sandpiles and random walks

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We show that the dissipative Abelian sandpile on a graph $\mathcal{L}$ can be related to a random walk on a graph that consists of $\mathcal{L}$ extended with a trapping site. From this relation it can be shown, using exact results and a scaling assumption, that the correlation length exponent ν of the dissipative sandpiles always equals $1/d_w$, where d_w is the fractal dimension of the random walker. This leads to a new understanding of the known result that $\nu=1/2$ on any Euclidean lattice. Our result is, however, more general, and as an example we also present exact data for finite Sierpinski gaskets, which fully confirm our predictions.

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Self-organized criticality (SOC) [1,2] is the phenomenon in which a slowly driven system with many interacting degrees of freedom evolves spontaneously into a critical state, characterized by long range correlations in space and time (for introductory reviews, see [3,4]). This phenomenon has by now been recognized (or conjectured to exist) in many (models of) natural phenomena such as earthquakes [5], forest fires [6], speciation of life [7], etc. Moreover, the possible presence of SOC can be investigated in experimentally controllable phenomena such as the Barkhausen effect [8], rice piles [9], and so on.

A question which is, however, still poorly understood is, What precisely are the necessary ingredients a system (or a model) must have for it to become self-organized critical? For example, one may ask whether or not the dissipation of "energy" (or a similar quantity) destroys long range correlations. This problem has been studied extensively in the Olami-Feder-Christensen (OFC) model of earthquakes [5]. Numerical studies originally seemed to show convincingly that even in the presence of a small amount of dissipation the model remains critical [10]. Later, it was shown exactly that at least in mean field, the OFC model is only critical when its energy is conserved [11]. Most recently it was argued, on the basis of a study of branching rates, that the same is true on a finite dimensional lattice [12].

The situation is less controversial for sandpile models which form *the* paradigmatic examples of systems showing SOC. In the past decade much progress has been made in the theoretical understanding of this type of models. This is especially true for the Abelian model [1,2], where, following the original work of Dhar [13], a mathematical formalism was developed [10–14] that allows an exact calculation of several properties of the model, such as height probabilities [14,15], wave properties [16,17], the upper critical dimension [18], and so on.

The role of conservation (of sand) in sandpile models was first studied numerically by Ghaffari *et al.* [19], who found that any amount of dissipation destroys the presence of SOC in the model. Recently, it was proven that indeed on any hypercubic lattice a nonconservative Abelian sandpile model is not critical [20]. There exists therefore a correlation length ξ in the system which diverges when the dissipation rate goes to zero. This allows the introduction of an exponent ν that describes this divergence. Numerically [19] it was found that $\nu\approx 1/2$ in $d=2$. The same authors argued on the basis of a renormalization group calculation that $\nu=1/2$ on any Euclidean lattice. This result was recently proven exactly [21].

In this Rapid Communication we study further the Abelian sandpile model with dissipation. We begin by showing that this problem can be related to that of a suitably defined random walker on a lattice with a trap. This result is quite general and extends an earlier mapping between conservative sandpiles and resistor networks (or equivalently the $q\rightarrow 0$ Potts model) [14]. Indeed, we will show that sending the dissipation to zero is equivalent to taking the long time limit of the random walk problem. It therefore comes as no surprise that the (sandpile) correlation length exponent ν can be related to the exponent $1/d_w$, which describes the asymptotic behavior of the random walker. Our result implies that $\nu=1/2$ for any Euclidean lattice. We thus recover in this situation the conclusion of Refs. [19,21] but add a new understanding of it. However, our prediction is more general and holds also on, for example, fractal lattices, or for certain types of random sandpiles. As an example, we performed calculations on the Sierpinski gasket for which d_w is known exactly ($d_w=\ln 5/\ln 2$). Our data are consistent with the prediction $\nu=1/d_w$.

The Abelian sandpile model on an arbitrary graph $\mathcal{L}$ (with N vertices) is defined as follows. On each vertex i of the graph, there exists a height variable z_i that assumes integer values and has the interpretation of energy or number of sand grains at site i. The dynamics of the model consists of two steps. First, we choose any site i at random, and add one grain of sand to that site, i.e., $z_i\rightarrow z_i+1$. When at a given site i, the height of sand exceeds a threshold z_{ic}, i.e. for $z_i>z_{ic}$, we say that that site becomes unstable and then the grains of sand on i are distributed among neighboring sites. This process, called *toppling*, is specified by a matrix Δ_{ij} such that

$$z_j\rightarrow z_j-\Delta_{ij}. \quad (1)$$

The elements Δ_{ij} satisfy $\Delta_{ii}>0$ and $\Delta_{ij}\leqslant 0$ when $i\neq j$, and the condition $\Sigma_j\Delta_{ij}\geqslant 0$, which guarantees that no sand is created in the toppling process. We will also limit ourselves to cases in which Δ is symmetric. Through toppling, neighboring sites can become unstable and in this way an *avalanche* of topplings is generated. A new grain of sand is added only when the avalanche is over, i.e., all sites are stable again. Finally, grains of sand can leave the system through certain boundary sites. These are necessary for the model to reach a steady state asymptotically. The number of topplings in a given avalanche s is a random variable whose distribution $P(s,N)$ is now known to have rich, multifractal properties [22]. In the present paper we will, however, only be interested in the first moment $\langle s\rangle$ of this distribution. An exact expression for this quantity can be obtained as follows [13]. One first introduces the matrix G, which is the inverse of Δ. The element G_{ij} can be interpreted as the expected number of topplings at site j when a grain of sand has been dropped at site i [13]. From this interpretation one obtains

$$\langle s\rangle=\frac{1}{N}\sum_{i\in\mathcal{L}}\sum_{j\in\mathcal{L}}G_{ij}. \tag{2}$$

One of the results of this paper will be a scaling expression for $\langle s\rangle$ for the dissipative sandpile model.

To continue we will reason further with the case of a graph where each site (apart from the boundary sites) is connected to a fixed number of neighbors z. We take $z_{ic}=z,\forall i$. For example, on the square lattice or on the Sierpinski gasket $z=4$. In the case of the conservative sandpile model we choose the matrix Δ as

$$\Delta^c_{i,j}=\begin{cases} z & \text{if } i=j\\ -1 & i \text{ and } j \text{ are neighbors}\\ 0 & \text{otherwise}\end{cases} \tag{3}$$

[in the rest of this paper a superscript c (respectively d) will refer to the conservative (respectively dissipative) case]. On the other hand, following Tsuchiya and Katori [20], we choose in the case of a dissipative sandpile model

$$\Delta^d_{i,j}=\begin{cases} z\gamma\zeta & \text{if } i=j\\ -\zeta & i \text{ and } j \text{ are neighbors}\\ 0 & \text{otherwise,}\end{cases} \tag{4}$$

with $\gamma>1$. In this way, at each toppling $\zeta z(\gamma-1)$ grains of sand disappear.

With each toppling matrix Δ_{ij} we can associate a random walk problem. To do this we have to extend the graph $\mathcal{L}$ with one extra site, denoted as T which, as we will see immediately, will get the properties of a trap for the random walker. Let $\mathcal{L}^\star=\mathcal{L}\cup T$. We then define a continuous time random walker on $\mathcal{L}^\star$ using the matrix elements Δ_{ij} as transition rates. More concrete, the rate by which the walker jumps from j to i is given by $-\Delta_{ij}$ for any two sites on $\mathcal{L}$. Second, the walker jumps from a site $j\in\mathcal{L}$ to the trap T with rate $\phi_j=\Sigma_i\Delta_{ij}$. Once the walker reaches T, it stays there forever.

Let $P(i,k,t)$ be the conditional probability that the walker is at site $i\in\mathcal{L}^\star$ at time t given that he was in k at $t=0$. This conditional probability then evolves according to the master equation

$$\dot{P}(i,k,t)=-\sum_j D_{ij}P(j,k,t), \tag{5}$$

where

$$D_{ij}=\begin{cases}\Delta_{ij} & i\in\mathcal{L},\ j\in\mathcal{L},\quad i\neq j\\ -\phi_j & i=T,\ j\in\mathcal{L}\\ 0 & \text{if } j=T.\end{cases} \tag{6}$$

For the diagonal elements we take

$$D_{jj}=-\sum_{i\in\mathcal{L}^\star,i\neq j}D_{ij}=-\sum_{i\in\mathcal{L},i\neq j}\Delta_{ij}+\phi_j=\Delta_{jj}.$$

In this way, and because of the conditions we put on the matrix Δ, D has all the necessary properties (see, e.g., [23]) of a stochastic matrix.

To solve master equation (5), it is common ([23]) to introduce the Green function $G_{ik}(s)$, which is the Laplace transform of $P(i,k,t)$,

$$G_{ik}(s)=\int_0^\infty P(i,k,t)e^{-st}dt. \tag{7}$$

The Green function obeys the linear equation

$$\delta_{ik}-sG_{ik}(s)=\sum_j D_{ij}G_{jk}(s)$$

whose formal solution is

$$G_{ik}(s)=\left(\frac{1}{s1+D}\right)_{ik}. \tag{8}$$

This solution can be written in terms of the eigenvalues λ_α and associated eigenvectors u_α of D as

$$G_{ik}(s)=\sum_\alpha\frac{1}{(s+\lambda_\alpha)}(u_\alpha)_i(u_\alpha)_k. \tag{9}$$

Notice that because of structure (6), the spectrum of the matrix consists of the spectrum of Δ and one zero eigenvalue, which we will denote as λ_0. The eigenvector associated with this zero eigenvalue is completely concentrated on the trap, $(u_0)_i=\delta_{i,T}$. On the other hand, for the eigenvectors associated with the other eigenvalues, we have $(u_\alpha)_T=0,\alpha\neq 0$. Therefore Eq. (9) becomes

$$G_{ik}(s)=\frac{1}{s}\delta_{i,T}\delta_{k,T}+\tilde{G}_{ik}(s), \tag{10}$$

where

$$\tilde{G}_{ik}(s)=\sum_{\alpha\neq 0}\frac{1}{(s+\lambda_\alpha)}(u_\alpha)_i(u_\alpha)_k. \tag{11}$$

We can therefore relate the matrix elements G_{ij} appearing in Eq. (2) to the elements of the Green function as

$$G_{ij}=\tilde{G}_{ij}(s=0). \tag{12}$$

We now have all the necessary ingredients to discuss the dissipative sandpiles whose toppling matrix is given in Eq. (4). Comparing Eq. (3) with Eq. (4), we immediately see that

$$\Delta_{ij}^{d}=\zeta\Delta_{ij}^{c}+z\zeta(\gamma-1)\delta_{ij}. \tag{13}$$

This simple relation implies that the eigenvectors of Δ^{d} and Δ^{c} are the same and that their eigenvalues are trivially related by a multiplicative and an additive constant. Therefore, we get for the inverse of Δ_{ij}^{d} using Eq. (12) and the general definition (11),

$$G_{ij}^{d}=\tilde{G}_{ij}^{d}(s=0)=\frac{1}{\zeta}\tilde{G}_{ij}^{c}(s=z(\gamma-1)). \tag{14}$$

This is our main result. It shows the relation between the dissipative sandpile model and the random walker associated with the *conservative* sandpile model but at $s=z(\gamma-1)$. Taking the conservative limit $\gamma\to 1$ then corresponds precisely to taking the limit $t\to\infty$ in the random walk problem. Since that asymptotic limit is determined by the scaling exponent d_w of the random walk, it can already be expected that the correlation length exponent ν is related to d_w.

The precise connection between the two exponents can be obtained as follows. We will calculate $\langle s\rangle$ as given in Eq. (2) for the dissipative sandpile defined in Eq. (4). Summing over *i*, using Eqs. (14) and (7) we get

$$\sum_{i\in\mathcal{L}}G_{ik}^{d}=\frac{1}{\zeta}\int_{0}^{\infty}P_{0}^{c}(k,t)e^{-z(\gamma-1)t}dt, \tag{15}$$

where $P_0^c(k,t)=\Sigma_{i\in\mathcal{L}}P(i,k,t)$ is the probability that the walker that started at *k* has not yet been trapped at time *t*. To calculate $P_0^c(k,t)$ it is important to first consider the type of random walk that we have to investigate. Because of relation (14), we have to work with the random walker associated with the matrix Δ^{c} of the conservative sandpile. The resulting random walker is therefore such that only boundary sites, which in the conservative case are the only ones where sand leaves the system, are connected with the trap. Random walks of this type, at least on an Euclidean lattice, are easy to study. We will come back to that later on.

First, we consider, however, the limit $N\to\infty$. In that limit a random walker starting on a typical site will not be trapped for any finite *t*, since only the boundary sites at infinity are connected with the trap. Therefore, one has $P_0^c(k,t)=1$. Then the integral in Eq. (15) can immediately be performed, and since the result does not depend on *k* we get

$$\langle s\rangle=\frac{1}{\zeta z(\gamma-1)}. \tag{16}$$

This relation was first derived on the $d=2$ square lattice in Ref. [20] but we now see that it is quite general. In fact, it shows that the dissipative sandpile is *never critical*.

We now turn back to the case that the number of sites on the graph is finite, in which case $P_0^c(k,t)$ will be a decreasing function of time. Explicit results for this quantity can be obtained on Euclidean lattices with elementary Fourier techniques. In one dimension one obtains, for example,

$$P_0^c(k,t)=\frac{2}{L+1}\sum_{j=1}^{L}\sum_{n=1}^{L}e^{-\omega_n t}\sin\left(\frac{n\pi k}{L+1}\right)\sin\left(\frac{n\pi j}{L+1}\right), \tag{17}$$

where $\omega_n=2[1-\cos(n\pi/L+1)]$. Similar results can easily be obtained in higher dimensions. In the scaling limit, $L\to\infty,t\to\infty$, it is easy to see that this probability is of the form

$$P_0^c(k,t)\approx\frac{1}{L}F\left(\frac{k}{L},\frac{t}{L^2}\right). \tag{18}$$

On any Euclidean lattice, the fractal dimension d_w of the type of random walker that we consider here equals 2. On the basis of exact results such as Eq. (18), and on general physical intuition, it can be expected that for the random walkers that are connected with the trap only through some boundary sites, $P_0^c(k,t)$ has in general the following scaling behavior:

$$P_0^c(k,t)\approx\frac{1}{L^{d_f}}H\left(\frac{k}{L},\frac{t}{L^{d_w}}\right), \tag{19}$$

with *H* some scaling function. Here d_f is the (fractal) dimension of the graph.

After inserting Eq. (19) in Eq. (15), making a suitable change of variables, and also performing the resulting sum over *k*, we finally get the following scaling form for $\langle s\rangle$:

$$\langle s\rangle\sim L^{d_w}R(L(\gamma-1)^{1/d_w}), \tag{20}$$

where *R* is a scaling function.

From Eq. (20), we see that for the conservative case, $\gamma=1$, and on an Euclidean lattice $\langle s\rangle\sim L^2$, which is an old result obtained by Dhar [13]. For $\gamma>1$, we conclude from Eq. (20) that the exponent ν that describes the crossover between dissipative and conservative sandpiles equals $1/d_w$.

On an Euclidean lattice, we recover in this way a result first determined with an approximate renormalization technique [19] and recently obtained exactly [21]. Our result (20) is, however, much more general. The relation $\nu=1/d_w$ should also hold on, e.g., fractal lattices. As an example, we checked this scaling on finite Sierpinski gasket. In particular, we considered Sierpinski gaskets of *n* generations with $n\leqslant 7$. For each of these we calculated the matrix G_{ij}^{d} by calculating the inverse of Δ_{ij}^{d} using a computer. This we did for several values of γ. From G^d we then calculated $\langle s\rangle$ using Eq. (2). Finally, we plotted $\langle s\rangle L^{-d_w}$ versus $L(\gamma-1)^{1/d_w}$. These numerically exact results are shown in Fig. 1. For the Sierpinski gasket $d_w=\ln 5/\ln 2$. As can be seen the agreement with the scaling prediction is excellent. Together with the known result for the square lattice case, these data give very strong support for the conjecture that $\nu=1/d_w$.

One may now ask how general our prediction for ν is, and whether one can imagine situations in which it does not hold

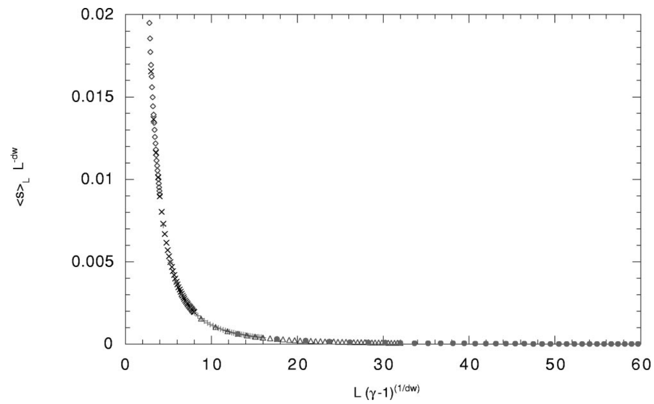

FIG. 1. Plot of $\langle s\rangle L^{-d_w}$ versus $L(\gamma-1)^{1/d_w}$ for the non conservative sandpile model on the Sierpinski gasket with $L=2,4,\ldots,128$, and different values of γ.

. Besides scaling assumption (19), the crucial step in the derivation is relation (13) between the conservative and dissipative toppling matrices. In a more general case one may consider a toppling matrix of the form

$$\Delta^d_{i,j}=\begin{cases} z\gamma_i\zeta & \text{if } \ i=j \\ -\zeta & i \ \text{ and } \ j \ \text{ are neighbors} \\ 0 & \text{otherwise,} \end{cases} \tag{21}$$

where γ_i is site dependent (the same would hold for a graph in which the coordination number is site dependent). Let $\gamma_m=\min_i\gamma_i$. For this case Eq. (14) gets replaced by

$$G^d_{ij}=\frac{1}{\zeta}\tilde{G}^r_{ij}(s=z(\gamma_m-1)),$$

where $\tilde{G}^r$ is the Green function of a random walker where from site i one enters the trap with rate $\zeta z(\gamma_i-\gamma_m)$. It then depends on the particular values of γ_i what is the precise value of d_w or even whether d_w can still be defined in a meaningful way. It is also not clear in which cases a scaling assumption such as Eq. (19) remains valid. But we expect that for many cases that are of interest from the sandpile point of view, our result will hold, eventually with an exponent ν that depends on the particular distribution of γ_i values.

As an example, take a matrix Δ^d_{ij} whose diagonal elements are constructed in the following random way. With probability p we take $\Delta^d_{ii}=z\zeta\gamma_1$, while with probability $1-p$ we take $\Delta^d_{ii}=z\zeta\gamma_2$. In the related random walker, and for $\gamma_2>\gamma_1$, the sites that are connected with the trap are randomly distributed over the lattice with probability p. For such a random walk, it has been shown that in $d=1$, $d_w=3$ [24]. Hence, we expect that for this kind of dissipative sandpile $\nu=1/3$ (instead of 1/2) in $d=1$. We are currently verifying this prediction. The results will be published elsewhere.

In summary, we have shown how to relate a dissipative sandpile model with an associated random walker. Using results from the theory of random walks we were then able to show that $\nu=1/d_w$ for a large class of nonconservative sandpile models. This result is in agreement with the available evidence on Euclidean lattices and on the Sierpinski gasket. Since the knowledge on random walks is quite extensive, we suspect that many interesting phenomena in dissipative sandpiles can now be obtained using the link with random walks.

Note added in proof. When this paper was under review, it was pointed out to us that the result $\nu=1/2$ for Euclidena lattices was also obtained using other approaches in Refs. [25] and [26].

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